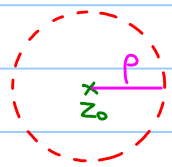
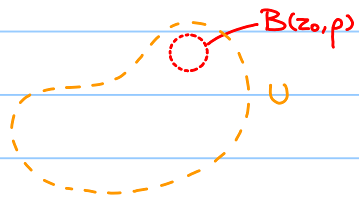
Terminologies :

(open) disk , centered at $z_0 \in \mathbb{C}$ with radius $\rho > 0$: $B(z_0, \rho) = \{ z \in \mathbb{C} : |z - z_0| < \rho \}$.



A subset $U \subseteq \mathbb{C}$ is open if $\forall z \in U$, $\exists \rho > 0$ s.t. $B(z_0, \rho) \subseteq U$.



U is said to be an open neighborhood of $z_0 \in \mathbb{C}$ if U is open and $z_0 \in U$.

A subset $V \subseteq \mathbb{C}$ is closed if $\mathbb{C} \setminus V$ is open.

A subset $D \subseteq \mathbb{C}$ is a domain if D is open and connected.

i.e. $\forall z_0, z_1 \in D$, $\exists \gamma : [0,1] \to D$ s.t.

(1) γ is cont.

(2) $\gamma(0) = z_0$, $\gamma(1) = z_1$

II) Derivatives

Let $f: \mathbb{C} \to \mathbb{C}$ be a function,

f is said to be differentiable at $z_0 \in \mathbb{C}$ if

$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists,

(OR rewrite as : $\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

which is denoted by $f'(z_0)$

e.g. Let $f(z) = z^n$, $n \in \mathbb{N}$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z}$$

Note : $(z+\Delta z)^n = z^n + C_1^n z^{n-1} \Delta z + C_2^n z^{n-2} (\Delta z)^2 + \cdots + (\Delta z)^n$

$$= \lim_{\Delta z \to 0} nz^{n-1} + \underbrace{C_2^n z^{n-2} \Delta z + \cdots + (\Delta z)^{n-1}}_{\text{terms involve } \Delta z}$$

$$= nz^{n-1}$$

$\therefore f'(z) = nz^{n-1}$ OR write $\frac{d}{dz} z^n = nz^{n-1}$

e.g. Let $f(z) = \frac{1}{z}$,

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\frac{1}{z+\Delta z} - \frac{1}{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-1}{z(z+\Delta z)}$$

$$= \frac{-1}{z^2}$$

e.g. Let $f(z) = \bar{z}$,

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(\bar{z} + \overline{\Delta z}) - \bar{z}}{\Delta z}$$

$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ does NOT exist.

Why ? Consider i) $\Delta z = \Delta x$ $\quad \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = 1$

ii) $\Delta z = i\Delta y$ $\quad \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = -1$

e.g. $f(z) = |z|^2 = |z|^2 + 0i$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \bar{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z}$$

exists only when $z = 0$

Note: $|z+\Delta z|^2 = (z+\Delta z)(\bar{z}+\overline{\Delta z})$

$= z\bar{z} + z\overline{\Delta z} + \bar{z}\Delta z + \Delta z\,\overline{\Delta z}$

$|z+\Delta z|^2 - |z|^2 = z\overline{\Delta z} + \bar{z}\Delta z + \Delta z\,\overline{\Delta z}$

Remarks:

1) f is differentiable at a certain point, but nowhere else in a neighborhood of that point.

2) If we write $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, then

$u(x,y) = x^2 + y^2$ and $v(x,y) = 0$.

They have continuous partial derivatives of all orders at every point.

i.e. Even $u(x,y)$ and $v(x,y)$ have continuous partial derivatives of all orders at a point, $f(z)$ may NOT differentiable at that point.

If f is differentiable at z_0, then f is continuous at z_0.

proof:

$$\lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} z - z_0 = f'(z_0) \cdot 0 = 0$$

$$\therefore \lim_{z \to z_0} f(z) = f(z_0)$$

Differentiation Formulas:

1) $(f \pm g)'(z) = f'(z) \pm g'(z)$

2) $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

3) $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$

proof of (2) :

$$\lim_{\Delta z\to 0} \frac{f(z+\Delta z)g(z+\Delta z)-f(z)g(z)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{f(z+\Delta z)g(z+\Delta z) - f(z+\Delta z)g(z) + f(z+\Delta z)g(z) - f(z)g(z)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} f(z+\Delta z)\cdot\frac{g(z+\Delta z)-g(z)}{\Delta z} + g(z)\cdot\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$= f(z)g'(z) + g(z)f'(z)$$

(f is diff at $z \Rightarrow f$ is cont. at z
$\Rightarrow \lim_{\Delta z\to 0} f(z+\Delta z) = f(z)$) $\blacksquare$

Theorem (Chain Rule)

Suppose that g is differentiable at z_0, and f is differentiable at $g(z_0)$.
Then $(f\circ g)(z) = f(g(z))$ is differentiable at z_0 and $(f\circ g)'(z_0) = f'(g(z_0))\,g'(z_0)$.

proof :

Assume $g'(z_0)\neq 0$, then $\underline{g(z)\neq g(z_0)}$ in a neighborhood of z_0.

$\Downarrow$ guarantee (*) works

$$\lim_{z\to z_0} \frac{f(g(z))-f(g(z_0))}{z-z_0}$$

$$= \lim_{z\to z_0} \frac{f(g(z))-f(g(z_0))}{g(z)-g(z_0)}\cdot\frac{g(z)-g(z_0)}{z-z_0} \quad \text{— (*)}$$

$$= f'(g(z_0))\cdot g'(z_0)$$

(g is diff at $z_0 \Rightarrow g$ is cont. at z_0
$\Rightarrow \lim_{z\to z_0} g(z)-g(z_0) = 0$)

When $g'(z_0)=0$,

f is diff. at $w_0 = g(z_0) \Rightarrow \left|\frac{f(w)-f(w_0)}{w-w_0}\right|$ is bounded in a nbd of w_0

$\left|\frac{f(w)-f(w_0)}{w-w_0}\right| \leq C$

$\Rightarrow |f(g(z))-f(g(z_0))| \leq C|w-w_0| = C|g(z)-g(z_0)|$ with $w=g(z)$

$\Rightarrow \underbrace{\left|\frac{f(g(z))-f(g(z_0))}{z-z_0}\right|}_{\downarrow\, 0} \leq C\underbrace{\left|\frac{g(z)-g(z_0)}{z-z_0}\right|}_{\downarrow\, 0}$ as $z\to z_0$

$\blacksquare$

e.g. $\frac{d}{dz}\frac{1}{z^2-1} = -\frac{2z}{(z^2-1)^2}$, $z\neq\pm 1$

Definition : A function $f(z)$ is analytic on the open set U if $f(z)$ is differentiable at each point of U and the complex derivative $f'(z)$ is continuous on U.

↑ In fact, automatically true, explain later.

If we say f is analytic at a point z_0, it means that f is analytic on an open set containing z_0.

III) Cauchy-Riemann Equations.

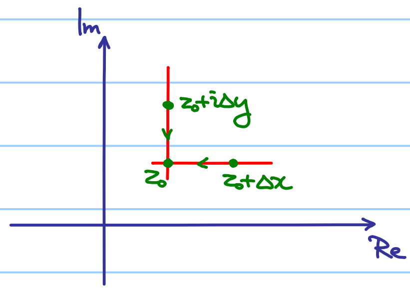
Suppose f is differentiable at $z_0 = x_0 + iy_0 \in \mathbb{C}$

$$f(x+iy) = u(x,y) + iv(x,y)$$

Any requirements on u and v at the point z_0 ?

We know $\lim\limits_{\Delta z \to 0} \dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ exists.

We consider two particular directions:



$$f'(z_0) = \lim_{\Delta x \to 0} \frac{f(z_0+\Delta x) - f(z_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0) + i(v(x_0+\Delta x, y_0) - v(x_0,y_0))}{\Delta x}$$

$$= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$$

Similarly,

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{f(z_0+i\Delta y) - f(z_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0) + i(v(x_0, y_0+\Delta y) - v(x_0,y_0))}{i\Delta y}$$

$$= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0)$$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ at $z_0 \in \mathbb{C}$ (called Cauchy-Riemann Equations)

$f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is differentiable at $z_0 \Rightarrow$ u, v satisfy CR-eq^n at z_0

$\overset{?}{\Leftarrow}$

Let $f(z) = \begin{cases} \frac{(\bar{z})^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$ $\qquad \frac{(\bar{z})^2}{z} = \frac{(x-iy)^2}{x+iy} = \frac{1}{x^2+y^2}[(x^3-3xy^2)+i(-3x^2y+y^3)]$

$u(x,y) = \begin{cases} \frac{x^3-3xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ $\qquad v(x,y) = \begin{cases} \frac{-3x^2y+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Ex: Show that u, v satisfy CR-eq^{n} at 0.

However, consider $\Delta x = \Delta y = t$ and $\Delta z = \Delta x + i\Delta y = t + it$

$$\lim_{t \to 0} \frac{f(0+(t+it)) - f(0)}{t+it} = -1$$

consider $\Delta z = \Delta x$,

$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = 1$$

$\therefore \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z}$ does NOT exist, i.e. f is NOT differentiable at 0.

"$\Leftarrow$" does NOT hold.

Theorem:

Suppose $f(z) = f(x+iy) = u(x,y) + i(x,y)$ is well-defined in some open neighborhood of $z_0 \in \mathbb{C}$ and the 1-st order partial derivatives of u and v exist in that neighborhood and continuous at (x_0, y_0).

If u, v satisfy CR-eq^{n} at (x_0, y_0), then $f'(z_0)$ exists.

e.g. Let $f(z) = e^z = e^x(\cos y + i \sin y)$

then $u(x,y) = e^x \cos y$, $v(x,y) = e^x \sin y$

Note: $u_x = v_y$ and $u_y = -v_x$ everywhere

and those partial derivatives are continuous everywhere

$\therefore f(z)$ is differentiable everywhere.

$f'(z) = u_x + i v_x = e^x(\cos y + i \sin y) = f(z)$.

Recall: $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$

e^z, iz $(-iz)$ are diff everywhere $\xRightarrow{\text{Chain Rule}}$ e^{iz} e^{-iz} is diff. everywhere $\frac{d}{dz}e^{iz} = ie^{iz}$

$\frac{d}{dz}e^{-iz} = -ie^{-iz}$

$\therefore \frac{d}{dz}\cos z = \dfrac{ie^{iz}-ie^{-iz}}{2} = -\sin z$

Ex: $\frac{d}{dz}\sin z = \cos z$

CR-eq^{n} in polar coordinates:

If $z_0 \neq 0$, then we consider an open neighborhood U of z_0 but $0 \notin U$.

$\forall z \in U$, z can be expressed as $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$f: U \subseteq \mathbb{C} \to \mathbb{C}$

$f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \qquad u_r = u_x\cos\theta + u_y\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \qquad u_\theta = -u_x r\sin\theta + u_y r\cos\theta$$

$$\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

if you like $\Downarrow$

$$v_r = v_x\cos\theta + v_y\sin\theta$$

$$v_\theta = -v_x r\sin\theta + v_y r\cos\theta$$

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u_r \\ u_\theta \end{pmatrix}$$

Now, $u_r = u_x\cos\theta + u_y\sin\theta = v_y\cos\theta - v_x\sin\theta = \frac{1}{r}v_\theta$

$u_\theta = -u_x r\sin\theta + u_y r\cos\theta = -v_y r\sin\theta - v_x r\cos\theta = -rv_r$

$\therefore$ CR-eq^{n} in polar coordinates:

$$u_r = \frac{1}{r}v_\theta, \quad u_\theta = -rv_r$$

$$f'(z) = u_x + iv_x = e^{i\theta}(u_r + iv_r)$$

e.g. Recall $\operatorname{Log}: \mathbb{C}\setminus(-\infty,0] \longrightarrow \mathbb{C}$

If $z \in \mathbb{C}\setminus(-\infty,0]$, $z = re^{i\theta}$ where $r>0$, $-\pi<\theta<\pi$

$\operatorname{Log} z = \log r + i\theta$

$\therefore u(r,\theta) = \log r$, $v(r,\theta) = \theta$

We have $u_r = \frac{1}{r}v_\theta$, $u_\theta = -rv_r$ $\Rightarrow$ Log is diff on $\mathbb{C}\setminus(-\infty,0]$

$f'(z) = u_x + iv_x = e^{i\theta}(u_r + iv_r) = \frac{1}{z}$

§ 3 Integrals

Motivation :

Fundamental Theorem of Calculus :

If $f:\mathbb{R}\to\mathbb{R}$ is a continuous function, then $F:\mathbb{R}\to\mathbb{R}$ defined by

$$F(x)=\int_{x_0}^{x} f(t)\,dt \quad \text{is a differentiable function and} \quad F'(x)=f(x)$$

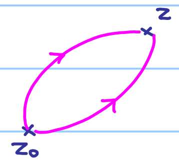
i.e. F is an antiderivative of f.

Question : Similar result in complex case ?

If $f:\mathbb{C}\to\mathbb{C}$ is a ?? function, then $F:\mathbb{C}\to\mathbb{C}$ defined by

$$F(z)=\int_{z_0}^{z} f(t)\,dt \quad \text{is a differentiable function and} \quad F'(z)=f(z)$$

trouble here ! How to define ?



there are infinitely many curves joining z_0 and z_1 !

I) Contours :

γ is said to be an arc if $\gamma: I\longrightarrow\mathbb{C}$ is a continuous function, where I is an interval.

If we write $\gamma(t)=x(t)+iy(t)$, where $x,y: I\longrightarrow\mathbb{R}$,

it just means that x and y are continuous.

Integral along $\gamma:[a,b]\longrightarrow\mathbb{C}$ is defined as follows :

$$\int_a^b \gamma(t)\,dt=\int_a^b x(t)\,dt+i\int_a^b y(t)\,dt$$

If $\alpha(t)=X(t)+iY(t)$ such that $\alpha'(t)=\gamma(t)$ $\forall t\in(a,b)$ and α is continuous on $[a,b]$

(i.e. $X'(t)=x(t)$, $Y'(t)=y(t)$)

$$\begin{aligned}\text{then } \int_a^b \gamma(t)\,dt &= \int_a^b x(t)\,dt+i\int_a^b y(t)\,dt\\ &=\big(X(b)-X(a)\big)+i\big(Y(b)-Y(a)\big)\\ &=\alpha(b)-\alpha(a)\end{aligned}$$

e.g. Note : $\frac{d}{dt}(-ie^{it})=e^{it}$

$$\int_0^{\frac{\pi}{4}} e^{it}\,dt=[-ie^{it}]_0^{\frac{\pi}{4}}=-ie^{i\frac{\pi}{4}}+i$$

Also, we have $\left|\int_a^b \gamma(t)\,dt\right| \le \int_a^b |\gamma(t)|\,dt$ (Estimate $\int_a^b \gamma(t)\,dt$ by a real integral)

It is trivial if $\int_a^b \gamma(t)\,dt = 0$.

Assume $\int_a^b \gamma(t)\,dt = r_0 e^{i\theta_0} \neq 0$

$$r_0 = \int_a^b e^{-i\theta_0}\gamma(t)\,dt$$
$$= \mathrm{Re}\int_a^b e^{-i\theta_0}\gamma(t)\,dt$$
$$= \int_a^b \mathrm{Re}(e^{-i\theta_0}\gamma(t))\,dt$$

Note: $\mathrm{Re}(e^{-i\theta_0}\gamma(t)) \le |e^{-i\theta_0}\gamma(t)| = |\gamma(t)|$

$$\le \int_a^b |\gamma(t)|\,dt$$

$\therefore \left|\int_a^b \gamma(t)\,dt\right| = r_0 \le \int_a^b |\gamma(t)|\,dt$.

An arc γ is said to be closed if $\gamma(a) = \gamma(b)$

An arc γ is said to be a simple arc, or a Jordan arc, if $\gamma(t_1) \neq \gamma(t_2)$ $\forall\, t_1 \neq t_2$

An arc $\gamma: [a,b] \to \mathbb{C}$ is said to be a simple closed curve, or a Jordan curve, if γ is simple except $\gamma(a) = \gamma(b)$.

An (open) arc $\gamma: (a,b) \longrightarrow \mathbb{C}$ is said to be differentiable if $x, y: (a,b) \longrightarrow \mathbb{R}$ are differentiable, we write $\gamma'(t) = x'(t) + iy'(t)$

furthermore, it is said to be regular if $\gamma'(t) \neq 0$ $\forall\, t \in (a,b)$.

Note: $\gamma'(t)$ = velocity, regular $\Rightarrow$ velocity is nonzero at every moment.

Suppose $\gamma: (a,b) \longrightarrow \mathbb{C}$ is a differentiable arc (parametrization)

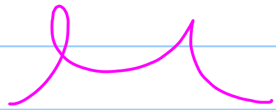
Since $|\gamma'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$,

Length of $\gamma = \int_a^b |z'(t)|\,dt$

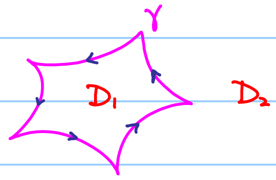
An arc $\gamma: [a,b] \longrightarrow \mathbb{C}$ is said to be a contour if

$\exists\; a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ s.t.

$\gamma|_{(t_{i-1}, t_i)}$ is regular and $\gamma'|_{(t_{i-1}, t_i)}$ is continuous for $i = 1, 2, \cdots, n$.



contour



Simple closed contour

Jordan Curve Theorem :

$\mathbb{C} \setminus \gamma = D_1 \sqcup D_2$ (disjoint union)

D_1, D_2 are domains , D_1 is bounded , D_2 is unbounded .

D_1 is simply connected ,

i.e. any loop in D_1 is contractiable to a point.

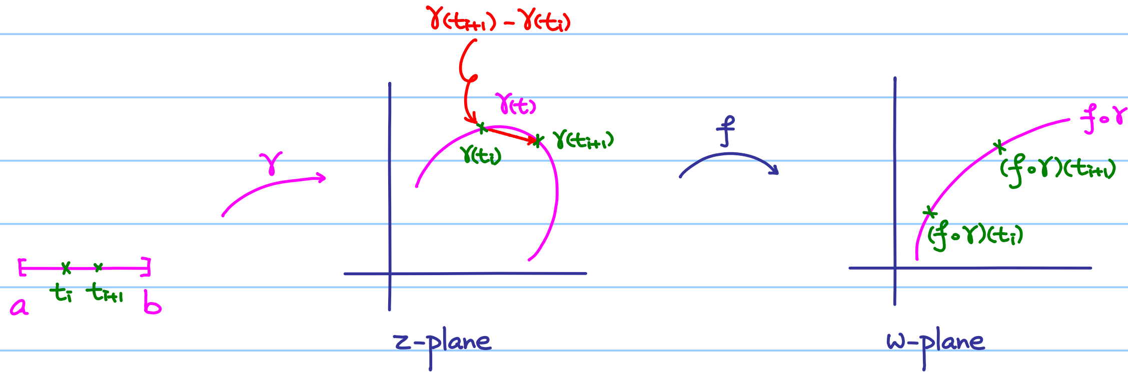
II) Contour Integrals

Suppose $\gamma : [a,b] \to \mathbb{C}$ is a contour, $\gamma(t) = x(t) + iy(t)$

$f : \mathbb{C} \to \mathbb{C}$ is a function, $f(z) = f(x+iy) = u(x,y) + iv(x,y)$

Then $(f \circ \gamma) : [a,b] \to \mathbb{C}$, and we write $w(t) = (f \circ \gamma)(t) = u(x(t), y(t)) + i\,v(x(t), y(t))$

$= u(t) + iv(t)$



Rough idea :

$$\sum_{i=1}^{n} f(z_i)\,\Delta z_i \qquad (\text{along } \gamma)$$

$$= \sum_{i=1}^{n} (f\circ\gamma)(t_i) \cdot (\gamma(t_{i+1}) - \gamma(t_i))$$

$$= \sum_{i=1}^{n} (f\circ\gamma)(t_i) \cdot \frac{\gamma(t_{i+1}) - \gamma(t_i)}{t_{i+1} - t_i} \, (t_{i+1} - t_i)$$

Taking limit $n \to \infty$ $\downarrow$

$$= \int_a^b (f\circ\gamma)(t) \cdot \gamma'(t)\, dt$$

Furthermore, suppose that $f\circ\gamma$ is piecewise continuous,

i.e. $u(t)$ and $v(t)$ are piecewise continuous, we define the contour integral of f along γ as follows : $\displaystyle \int_\gamma f(z)\,dz = \int_a^b (f\circ\gamma)(t) \cdot \gamma'(t)\, dt$

e.g. $f(z) = \frac{1}{z}$, $\gamma(t) = \cos t + i \sin t = e^{it}$ $\quad 0 \le t \le 2\pi$ $\quad$ (unit circle)

$\gamma'(t) = ie^{it}$

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} e^{-it} \cdot ie^{it}\, dt$$

$$= \int_0^{2\pi} i\, dt$$

$$= 2\pi i$$

Ex: $f(z)=\frac{1}{z}$, $\alpha(t)=\cos 2\theta + i\sin 2\theta = e^{2i\theta}$ $\quad 0\le\theta\le\pi$ (unit circle)

$\int_\alpha f(z)\,dz = ?$ Ans: $2\pi i$

Remark: Contour integral is independent from choice of parameterization of the contour.

e.g. $f(z)=\frac{1}{z^2}$, $\gamma(t)=\cos t + i\sin t = e^{it}$ $\quad 0\le t\le 2\pi$ (unit circle)

$\gamma'(t) = ie^{it}$

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_0^{2\pi} e^{-2it}\cdot ie^{it}\,dt \\ &= \int_0^{2\pi} ie^{-it}\,dt \\ &= \left[-ie^{-it}\right]_0^{2\pi} \\ &= 0\end{aligned}$$

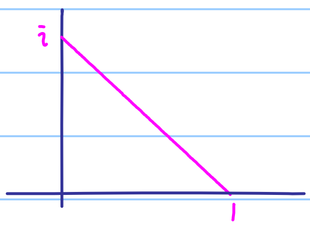
Suppose $\exists M>0$ s.t. $|f(\gamma(t))|\le M \quad \forall t\in[a,b]$, then we have the following estimate

ML-estimate :

$$\begin{aligned}\left|\int_\gamma f(z)\,dz\right| &= \left|\int_a^b (f\circ\gamma)(t)\cdot\gamma'(t)\,dt\right| \\ &\le \int_a^b |(f\circ\gamma)(t)\cdot\gamma'(t)|\,dt \\ &= \int_a^b |f\circ\gamma(t)|\cdot|\gamma'(t)|\,dt \\ &\le M\cdot\int_a^b |\gamma'(t)|\,dt \\ &= ML\end{aligned}$$

e.g. Let γ denote the line segment from i to 1.

Using ML-estimate to estimate $\left|\int_\gamma \frac{dz}{z^4}\right|$



Note: $|z|$ attains min when $z=\frac{1}{2}(1+i)$. (Why?)

$\therefore\ |z|\ge\frac{1}{\sqrt2}$

$\left|\frac{1}{z^4}\right|\le 2 \quad \forall z\in\gamma \qquad M=2$

$L=\sqrt{1^2+1^2}=\sqrt2$

$\therefore\ \left|\int_\gamma \frac{dz}{z^4}\right| \le ML = 2\sqrt2$

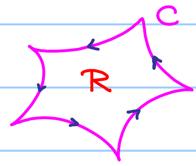
III) Cauchy-Goursat Theorem

Suppose C is a simple closed contour which is parametrized in positive sense (counterclockwise)

R is the bounded domain bounded by C.

so $C = \partial R$ = boundary of R

Let $\bar{R} = R \cup \partial R$



Suppose f is analytic on $\bar{R}$

i.e. analytic in an open neighborhood containing $\bar{R}$

f has continuous derivative

We write $f(z) = u(x,y) + iv(x,y)$, $\gamma(t) = x(t) + iy(t)$

Then
$$\int_\gamma f(z)dz = \int_a^b (f\circ\gamma)(t)\cdot\gamma'(t)dt$$
$$= \int_a^b (u(t) + iv(t))\cdot(x'(t) + iy'(t))$$
$$= \int_a^b (ux' - vy') + i(vx' + uy')\, dt$$

① Green Theorem

$P(x,y)$, $Q(x,y)$

P_x, P_y, Q_x, Q_y continuous on $\bar{R}$

$\Rightarrow \int_\gamma Pdx + Qdy = \iint_R (Q_x - P_y)\,dA$

$$\therefore \int_\gamma f(z)dz = \int_\gamma udx - vdy + i\int_\gamma vdx + udy$$
(*)
$$= \iint_R (-v_x - u_y)dA + i\iint_R (u_x - v_y)dA$$
$$= 0$$

② CR-eq^n

$u_x = v_y$, $u_y = -v_x$

(*) This step needs the continuity of u_x, u_y, v_x, v_y

It can be guaranteed by the analyticity of f,

in particular, the continuity of f'.

In fact, we have the same result without the assumption of the continuity of f'.

Cauchy-Goursat Theorem :

If a function f is differentiable at all points of $\bar{R}$ (i.e. an open neighborhood of $\bar{R}$)

then $\int_\gamma f(z)dz = 0$.

e.g. $f(z) = z$, $\gamma(t) = \cos t + i\sin t = e^{it}$ $\quad 0 \le t \le 2\pi$ (unit circle)

$\gamma'(t) = ie^{it}$

$\int_\gamma f(z)\,dz = 0$ since $f(z) = z$ is differentiable everywhere.

Verify: $\int_\gamma f(z)\,dz = \int_0^{2\pi} e^{it} \cdot ie^{it}\,dt$

$= \int_0^{2\pi} ie^{2it}\,dt$

$= \left[\frac{1}{2}e^{2it}\right]_0^{2\pi}$

$= 0$

Ex: Let $f(z) = \frac{1}{z}$, γ be the circle centered at 2 with radius 1.

Verify $\int_\gamma f(z)\,dz = 0$ by explicit computation.

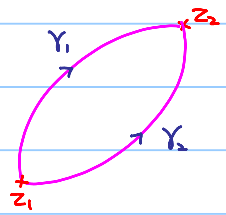
Theorem:

Suppose f is continuous on a domain D (NOT necessary to be simply connected)

The following statements are equivalent:

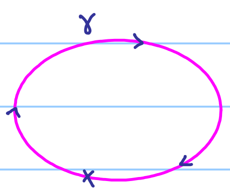
1) f has an antiderivative F in D;

2) the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed points z_1 to any fixed point z_2 all have the same value;



$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$

3) the integrals of $f(z)$ along any closed contours lying entirely in D all have value zero.



$\int_\gamma f(z)\,dz = 0$.